

# Double hadron lepto-production in the current and target fragmentation regions

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## ABSTRACT

We study the inclusive production of two hadrons in deep inelastic processes,  $\ell N \rightarrow \ell h_1 h_2 X$ , with  $h_1$  in the current fragmentation region (CFR) and  $h_2$  in the target fragmentation region (TFR). Assuming a factorized scheme, the recently introduced polarized and transverse momentum dependent fracture functions couple to the transverse momentum dependent fragmentation functions. This allows the full exploration of the fracture functions for transversely polarized quarks. Some particular cases are considered.

*Key words:* Semi-inclusive DIS, Current Fragmentation, Target Fragmentation, Fracture Functions, Fragmentation Functions, Polarization, Transverse Momentum

# 1 Introduction

In a recent paper [1] we have introduced the formalism of polarized and transverse momentum dependent fracture functions to describe semi-inclusive deep inelastic scattering in the target fragmentation region. We have shown that, at leading order of QCD, considering only the production of spinless or unpolarized hadrons, there are 16 independent fracture functions, which describe the conditional probabilities of finding unpolarized or polarized quarks inside unpolarized or polarized nucleons fragmenting into the final observed hadron. We have also given explicit sum rules which, upon integration over the momentum of the final hadron in the TFR, relate the fracture functions to the usual transverse momentum dependent distribution functions (TMDs).

The 16 fracture functions can be divided into three classes, referring respectively to unpolarized (4), longitudinally polarized (4) and transversely polarized (8) quarks. The first 8 can be accessed in single-hadron or single-hadron + jet production: explicit expressions of the corresponding cross sections have been derived and interesting azimuthal dependences, which have to be compared with those found for single hadrons produced in the CFR, have been discussed [1].

The 8 fracture functions related to transversely polarized quarks are chiral-odd quantities and can only appear in physical observables containing another chiral-odd function. This can be achieved by considering the combined production of two hadrons, one in the TFR and one in the CFR; the production of the latter is described by transverse momentum dependent fragmentation functions and the Collins mechanism provides the necessary chiral-odd quantity. We develop here the full formalism for such a double hadron lepto-production, following and completing the work of Ref. [1].

## 2 Double hadron lepto-production

Let us start from a general two-particle inclusive lepto-production process

$$l(\ell) + N(P) \rightarrow l(\ell') + h_1(P_1) + h_2(P_2) + X(P_X).$$

In the one-photon exchange approximation its cross section reads

$$d\sigma = \frac{1}{4\ell \cdot P} \frac{e^4}{Q^4} L_{\mu\nu} W^{\mu\nu} (2\pi)^4 \frac{d^3\ell'}{(2\pi)^3 2E'} \frac{d^3\mathbf{P}_1}{(2\pi)^3 2E_1} \frac{d^3\mathbf{P}_2}{(2\pi)^3 2E_2}, \quad (1)$$

where  $L^{\mu\nu}$  is the ordinary leptonic tensor and  $W^{\mu\nu}$  is the hadronic tensor:

$$\begin{aligned} W^{\mu\nu} &= \frac{1}{(2\pi)^4} \sum_a e_a^2 \sum_X \int \frac{d^3P_X}{(2\pi)^3 2E_X} (2\pi)^4 \delta^4(q + P - P_X - P_1 - P_2) \\ &\times \langle P, S | J^\mu(0) | P_1, P_2; X \rangle \langle P_1, P_2; X | J^\nu(0) | P, S \rangle. \end{aligned} \quad (2)$$

We are interested in two-particle inclusive lepto-production with one hadron ( $h_1$ ) in the CFR and one hadron ( $h_2$ ) in the TFR, as shown in Fig. 1. We assume that both hadrons are unpolarized or spinless.

Semi-inclusive DIS is usually described in terms of the three variables

$$x_B = \frac{Q^2}{2P \cdot q} \quad y = \frac{P \cdot q}{P \cdot \ell} \quad z_1 = \frac{P \cdot P_1}{P \cdot q}. \quad (3)$$

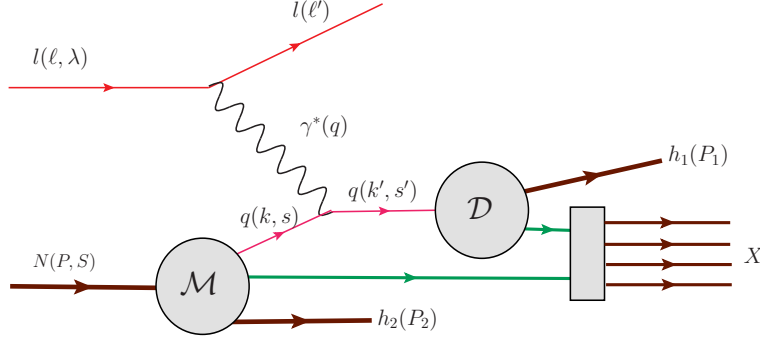


Figure 1: Lepto-production of two hadrons, one in the CFR and one in the TFR.

When a second hadron,  $h_2$ , is produced, one needs a further variable related to  $P_2$ . It is convenient to use a light-cone parametrization of vectors. Given a generic vector  $A^\mu = (A^0, A^1, A^2, A^3)$ , their light-cone components are defined as  $A^\pm \equiv (A^0 \pm A^3)/\sqrt{2}$  and we write  $A^\mu = [A^+, A^-, \mathbf{A}_\perp]$ .

We now introduce two null vectors,  $n_+^\mu = [1, 0, \mathbf{0}_\perp]$  and  $n_-^\mu = [0, 1, \mathbf{0}_\perp]$ , with  $n_+ \cdot n_- = 1$ , so that a vector can be parametrized as  $A^\mu = A^+ n_+^\mu + A^- n_-^\mu + A_\perp^\mu$ . We work in a frame where the target nucleon and the virtual photon are collinear (we call it a “ $\gamma^* N$  collinear frame”). The nucleon is supposed to move along the  $-z$  direction.

The unit vector  $\hat{\mathbf{q}} \equiv \mathbf{q}/|\mathbf{q}|$  identifies the positive  $z$  direction. In terms of the null vectors  $n_+^\mu$  and  $n_-^\mu$  the four-momenta at hand are (approximate equalities are valid up to terms proportional to some mass or transverse momentum squared):

$$P^\mu = P^- n_-^\mu + \frac{m_N^2}{2P^-} n_+^\mu \simeq P^- n_-^\mu, \quad (4)$$

$$q^\mu \simeq \frac{Q^2}{2x_B P^-} n_+^\mu - x_B P^- n_-^\mu, \quad (5)$$

$$P_1^\mu \simeq \frac{z_1 Q^2}{2x_B P^-} n_+^\mu + \frac{(\mathbf{P}_{1\perp}^2 + m_1^2)x_B P^-}{z_1 Q^2} n_-^\mu + P_{1\perp}^\mu \simeq \frac{z_1 Q^2}{2x_B P^-} n_+^\mu + P_{1\perp}^\mu, \quad (6)$$

$$P_2^\mu = \zeta_2 P^+ n_+^\mu + \frac{\mathbf{P}_{2\perp}^2 + m_2^2}{2\zeta P^+} n_-^\mu + P_{2\perp}^\mu \simeq \zeta_2 P^+ n_+^\mu + P_{2\perp}^\mu. \quad (7)$$

Replacing  $\mathbf{P}_1$  and  $\mathbf{P}_2$  with the variables  $(z_1, \mathbf{P}_{1\perp})$  and  $(\zeta_2, \mathbf{P}_{2\perp})$  respectively, the cross section takes the form

$$\frac{d\sigma}{dx_B dy dz_1 d\zeta_2 d^2\mathbf{P}_{1\perp} d^2\mathbf{P}_{2\perp} d\phi_S} = \frac{\alpha_{\text{em}}^2}{8(2\pi)^3 Q^4} \frac{y}{z_1 \zeta_2} L_{\mu\nu} W^{\mu\nu}. \quad (8)$$

Here  $\phi_S$  is the azimuthal angle of the transverse component of  $S^\mu$ , the nucleon spin vector, parametrized as

$$S^\mu = S_\parallel \frac{P^- n_-^\mu}{m_N} - S_\parallel \frac{m_N}{2P^-} n_+^\mu + S_\perp^\mu \simeq S_\parallel \frac{P^- n_-^\mu}{m_N} + S_\perp^\mu. \quad (9)$$

The explicit expression of the symmetric part of the leptonic tensor in the  $\gamma^* N$  collinear frame

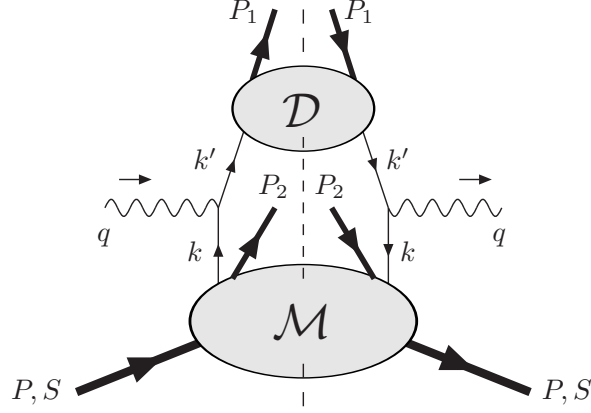


Figure 2: The handbag diagram for double hadron lepto-production.

is [2]

$$L_{(s)}^{\mu\nu} = \frac{Q^2}{y^2} \left\{ -2 \left( 1 - y + \frac{y^2}{2} \right) g_{\perp}^{\mu\nu} + 4(1 - y) \left[ \frac{x_B^2 (P^-)^2}{Q^2} n_-^\mu n_-^\nu + \frac{Q^2}{4x_B^2 (P^-)^2} n_+^\mu n_+^\nu + \frac{1}{2} n_-^{\{\mu} n_+^{\nu\}} \right] \right. \\ \left. + 4(1 - y) \left( \hat{\ell}_\perp^\mu \hat{\ell}_\perp^\nu + \frac{1}{2} g_{\perp}^{\mu\nu} \right) + 2(2 - y) \sqrt{1 - y} \left[ \frac{x_B P^-}{Q} n_-^{\{\mu} \hat{\ell}_\perp^{\nu\}} + \frac{Q}{2x_B P^-} n_+^{\{\mu} \hat{\ell}_\perp^{\nu\}} \right] \right\}, \quad (10)$$

where  $\ell_\perp^\mu$  is the transverse component of the incoming and outgoing lepton momentum ( $\hat{\ell}_\perp^\mu = \ell_\perp^\mu / |\ell_\perp|$ ), and  $g_{\perp}^{\mu\nu} = g^{\mu\nu} - (n_+^\mu n_-^\nu + n_-^\mu n_+^\nu)$ . The antisymmetric part of the leptonic tensor reads ( $\lambda$  is the helicity of the lepton and  $\epsilon_{\perp}^{\mu\nu} \equiv \epsilon^{\mu\nu}_{\rho\sigma} n_-^\rho n_+^\sigma$ )

$$L_{(a)}^{\mu\nu} = \frac{Q^2}{y^2} \left\{ -i \lambda y (2 - y) \epsilon_{\perp}^{\mu\nu} - 2i \lambda y \sqrt{1 - y} \epsilon^{\mu\nu}_{\rho\sigma} \left( \frac{x_B P^-}{Q} n_-^\rho - \frac{Q}{2x_B P^-} n_+^\rho \right) \hat{\ell}_{\perp\sigma} \right\}. \quad (11)$$

In the parton model, or equivalently at lowest order in QCD, the hadronic tensor for the associated hadron production in the current and the target fragmentation regions is represented by the handbag diagram of Fig. 2 and reads (to simplify the presentation, we consider only quarks, the extension to antiquarks being straightforward):

$$W^{\mu\nu} = \frac{1}{(2\pi)^4} \sum_a e_a^2 \sum_X \int \frac{d^3 \mathbf{P}_X}{(2\pi)^3 2E_X} \sum_{X'} \int \frac{d^3 \mathbf{P}'_X}{(2\pi)^3 2E'_X} \int \frac{d^4 k}{(2\pi)^4} \int \frac{d^4 k'}{(2\pi)^4} \times \\ [\bar{\chi}(k', P_1) \gamma^\mu \phi(k, P, P_2)]^* [\bar{\chi}(k', P_1) \gamma^\nu \phi(k, P, P_2)] \times \\ (2\pi)^4 \delta^4(P - k - P_2 - P_X) (2\pi)^4 \delta^4(k + q - k') (2\pi)^4 \delta^4(k' - P_1 - P'_X), \quad (12)$$

where  $\chi$  and  $\phi$  are matrix elements of the quark field  $\psi$  defined as

$$\chi(k', P_1) = \langle 0 | \psi(0) | P_1; X' \rangle, \quad (13)$$

$$\phi(k, P, P_2) = \langle P_2; X | \psi(0) | P, S \rangle. \quad (14)$$

We now introduce the fracture matrix  $\mathcal{M}$  representing the partonic structure of the nucleon target when it fragments into the final-state hadron  $h_2$ :

$$\begin{aligned} \mathcal{M}_{ij}(k; P, S; P_2) &= \sum_X \int \frac{d^3 \mathbf{P}_X}{(2\pi)^3 2E_X} \int \frac{d^4 \xi}{(2\pi)^4} e^{ik \cdot \xi} \times \\ &\quad \langle P, S | \bar{\psi}_j(0) | P_2; X \rangle \langle P_2; X | \psi_i(\xi) | P, S \rangle, \end{aligned} \quad (15)$$

and the fragmentation matrix  $\mathcal{D}$  representing the production of the hadron  $h_1$  from the current jet:

$$\mathcal{D}_{ij}(k'; P, S; P_1) = \sum_{X'} \int \frac{d^3 \mathbf{P}'_{X'}}{(2\pi)^3 2E'_{X'}} \int \frac{d^4 \eta}{(2\pi)^4} e^{ik' \cdot \eta} \langle 0 | \psi_i(\eta) | P_1; X' \rangle \langle P_1; X' | \bar{\psi}_j(0) | 0 \rangle. \quad (16)$$

In QCD, Wilson lines connecting the quark fields must be inserted in order to ensure gauge invariance.

Using the definitions (15, 16), the hadronic tensor becomes

$$W^{\mu\nu} = \sum_a e_a^2 \int d^4 k \int d^4 k' \delta^4(k + q - k') \text{Tr} [\mathcal{M} \gamma^\mu \mathcal{D} \gamma^\nu]. \quad (17)$$

In the parton model description of partly inclusive leptonproduction it is convenient to use another class of frames, where the hadron produced in the CFR is collinear with the target nucleon (we call them “ $hN$  collinear frames”). In these frames the virtual photon acquires a transverse momentum  $\mathbf{q}_T^\mu$  (we use  $T$  subscripts to denote transverse quantities in a  $hN$  collinear frame). The parametrizations of  $q^\mu$  and  $P_1^\mu$  then become

$$q^\mu \simeq \frac{Q^2}{2x_B P^-} n_+^\mu - x_B P^- n_-^\mu + q_T^\mu, \quad (18)$$

$$P_1^\mu \simeq \frac{z_1 Q^2}{2x_B P^-} n_+^\mu + \frac{(\mathbf{P}_{1\perp}^2 + m_1^2) x_B P^-}{z_1 Q^2} n_-^\mu \simeq \frac{z_1 Q^2}{2x_B P^-} n_+^\mu. \quad (19)$$

One can easily check that the relation between  $\mathbf{q}_T$  and  $\mathbf{P}_{1\perp}$  is  $\mathbf{q}_T = -\mathbf{P}_{1\perp}/z_1$ .

The quark momenta are parametrized as

$$k^\mu = x P^- n_-^\mu + \frac{\mathbf{k}_T^2 + k^2}{2x P^-} n_+^\mu + k_T^\mu \simeq x P^- n_-^\mu + k_T^\mu, \quad (20)$$

$$k'^\mu = \frac{P_1^+}{z} n_+^\mu + \frac{z(k'^2 + \mathbf{k}'_T^2)}{2P_1^+} n_-^\mu + k'_T{}^\mu \simeq \frac{P_1^+}{z} n_+^\mu + k'_T{}^\mu. \quad (21)$$

Here  $x$  is the fraction of the light-cone momentum of the target carried by the emitted quark, and  $z$  is the fraction of the light-cone momentum of the struck quark carried by the hadron in the CFR.

The delta function in Eq. (17) enforces the constraints  $x = x_B$  and  $z = z_1$ . Written explicitly, the hadronic tensor is

$$W^{\mu\nu} = \sum_a e_a^2 \int dk^+ dk^- d^2\mathbf{k}_T \int dk'^+ dk'^- d^2\mathbf{k}'_T \times \delta(k'^+ - P_1^+/z_1) \delta(k^- - x_B P^-) \delta^2(\mathbf{k}_T + \mathbf{q}_T - \mathbf{k}'_T) \text{Tr} [\mathcal{M} \gamma^\mu \mathcal{D} \gamma^\nu]. \quad (22)$$

Notice that while one should in principle distinguish between transverse vectors in a  $\gamma^* N$  collinear frame (labelled by a  $\perp$  subscript) and transverse vectors in a  $hN$  collinear frame (labelled by a  $T$  subscript), the difference is of order  $(P^-)^2$  and can be ignored as far as one neglects subleading corrections in  $P^-$  (i.e., higher twists).

### 3 Transverse momentum dependent fracture functions and fragmentation functions

The most general decomposition of the fracture matrix  $\mathcal{M}$  in a basis of Dirac matrices would contain terms proportional to  $\mathbb{1}, \gamma^\mu, \gamma^\mu \gamma_5, \gamma_5, \sigma^{\mu\nu} \gamma_5$ . We are interested in leading-twist fracture functions, i.e. in terms of  $\mathcal{M}$  that are of order  $(P^-)^1$ . At this order, only the vector, axial and tensor components of  $\mathcal{M}$  appear [3]:

$$\mathcal{M} = \frac{1}{2} (\mathcal{V}_\mu \gamma^\mu + \mathcal{A}_\mu \gamma_5 \gamma^\mu + i \mathcal{T}_{\mu\nu} \sigma^{\mu\nu} \gamma_5), \quad (23)$$

where the coefficients  $\mathcal{V}^\mu, \mathcal{A}^\mu$  and  $\mathcal{T}^{\mu\nu}$  contain various combinations of the vectors, or pseudo-vectors,  $P^\mu, P_1^\mu, P_2^\mu, k^\mu, k'^\mu$  and  $S^\mu$ .

The polarized transverse-momentum dependent fracture functions appear in the expansion of the leading twist Dirac projections ( $\Gamma = \gamma^-, \gamma^- \gamma_5, i\sigma^{i-} \gamma_5$ )

$$\begin{aligned} \mathcal{M}^{[\Gamma]}(x_B, \mathbf{k}_\perp, \zeta_2, \mathbf{P}_{2\perp}) &= \frac{1}{4\zeta_2} \int \frac{dk^+ dk^-}{(2\pi)^3} \delta(k^- - x_B P^-) \text{Tr} (\mathcal{M} \Gamma) \\ &= \frac{1}{4\zeta_2} \int \frac{d\xi^+ d^2\boldsymbol{\xi}_\perp}{(2\pi)^6} e^{i(x_B P^- \xi^+ - \mathbf{k}_\perp \cdot \boldsymbol{\xi}_\perp)} \sum_X \int \frac{d^3\mathbf{P}_X}{(2\pi)^3 2E_X} \times \\ &\quad \langle P, S | \bar{\psi}(0) \Gamma | P_2; X \rangle \langle P_2; X | \psi(\xi^+, 0, \boldsymbol{\xi}_\perp) | P, S \rangle. \end{aligned} \quad (24)$$

These represent the conditional probabilities to find an unpolarized ( $\Gamma = \gamma^-$ ), a longitudinally polarized ( $\Gamma = \gamma^- \gamma_5$ ) or a transversely polarized ( $\Gamma = i\sigma^{i-} \gamma_5$ ) quark with longitudinal momentum fraction  $x_B$  and transverse momentum  $\mathbf{k}_\perp$  inside a nucleon fragmenting into a hadron carrying a fraction  $\zeta_2$  of the nucleon longitudinal momentum and a transverse momentum  $\mathbf{P}_{2\perp}$ . Again, in QCD a Wilson line  $\mathcal{W}$  must be inserted, which for  $\mathbf{k}_\perp$ -dependent distributions includes transverse links and is generally rather complicated [4, 5]: its explicit structure, however, is irrelevant for our purposes.

The most general parameterization of the traced fracture matrix (24) at leading twist is:

$$\mathcal{M}^{[\gamma^-]} = \hat{u}_1 + \frac{\mathbf{P}_{2\perp} \times \mathbf{S}_\perp}{m_2} \hat{u}_{1T}^h + \frac{\mathbf{k}_\perp \times \mathbf{S}_\perp}{m_N} \hat{u}_{1T}^\perp + \frac{S_\parallel (\mathbf{k}_\perp \times \mathbf{P}_{2\perp})}{m_N m_2} \hat{u}_{1L}^h \quad (25)$$

$$\mathcal{M}^{[\gamma^-\gamma_5]} = S_{\parallel} \hat{l}_{1L} + \frac{\mathbf{P}_{2\perp} \cdot \mathbf{S}_{\perp}}{m_2} \hat{l}_{1T}^h + \frac{\mathbf{k}_{\perp} \cdot \mathbf{S}_{\perp}}{m_N} \hat{l}_{1T}^{\perp} + \frac{\mathbf{k}_{\perp} \times \mathbf{P}_{2\perp}}{m_N m_2} \hat{l}_1^{\perp h} \quad (26)$$

$$\begin{aligned} \mathcal{M}^{[i\sigma^{i-}\gamma_5]} &= S_{\perp}^i \hat{t}_{1T} + \frac{S_{\parallel} P_{2\perp}^i}{m_2} \hat{t}_{1L}^h + \frac{S_{\parallel} k_{\perp}^i}{m_N} \hat{t}_{1L}^{\perp} \\ &\quad + \frac{(\mathbf{P}_{2\perp} \cdot \mathbf{S}_{\perp}) P_{2\perp}^i}{m_2^2} \hat{t}_{1T}^{hh} + \frac{(\mathbf{k}_{\perp} \cdot \mathbf{S}_{\perp}) k_{\perp}^i}{m_N^2} \hat{t}_{1T}^{\perp\perp} \\ &\quad + \frac{(\mathbf{k}_{\perp} \cdot \mathbf{S}_{\perp}) P_{2\perp}^i - (\mathbf{P}_{2\perp} \cdot \mathbf{S}_{\perp}) k_{\perp}^i}{m_N m_2} \hat{t}_{1T}^{\perp h} \\ &\quad + \frac{\epsilon_{\perp}^{ij} P_{2\perp j}}{m_2} \hat{t}_1^h + \frac{\epsilon_{\perp}^{ij} k_{\perp j}}{m_N} \hat{t}_1^{\perp}, \end{aligned} \quad (27)$$

where by the vector product of the two-dimensional vectors we mean the pseudo-scalar quantity  $\mathbf{a}_{\perp} \times \mathbf{b}_{\perp} = \epsilon_{\perp ij} a_{\perp}^i b_{\perp}^j = |\mathbf{a}_{\perp}| |\mathbf{b}_{\perp}| \sin(\phi_b - \phi_a)$ . All fracture functions depend on the scalar variables  $x_B, \mathbf{k}_{\perp}^2, \zeta_2, \mathbf{P}_{2\perp}^2, \mathbf{k}_{\perp} \cdot \mathbf{P}_{2\perp}$ .

Notice that, with respect to Ref. [1], we have adopted a new, Amsterdam-style, nomenclature for fracture functions, which makes their correspondence with distribution functions more visible. In particular, we denote by  $\hat{u}$  (formerly,  $\hat{M}$ ) the unintegrated fracture functions of unpolarized quarks, by  $\hat{l}$  (formerly,  $\Delta \hat{M}$ ) the unintegrated fracture functions of longitudinally polarized quarks, and by  $\hat{t}$  (formerly,  $\Delta_T \hat{M}$ ) the unintegrated fracture functions of transversely polarized quarks. The subscript 1 denotes leading-twist quantities. The subscripts  $L$  and  $T$  label the polarization of the target (no subscript = unpolarized,  $L$  = longitudinally polarized,  $T$  = transversely polarized). The superscripts  $h$  and  $\perp$  signal the presence of factors  $P_{2\perp}^i$  and  $k_{\perp}^i$ , respectively. Fracture functions integrated over the hadron transverse momentum will have no hat; fracture functions integrated over the quark transverse momentum will have a tilde.

An important point to stress is that while parity invariance constrains the structure of the fracture matrix, time reversal invariance does not, since  $\mathcal{M}$ , similarly to the fragmentation matrix, contains the out-states  $|P_2; X\rangle$ .

Turning now to the fragmentation matrix, its Dirac projections are defined as

$$\begin{aligned} \mathcal{D}^{[\Gamma]}(z_1, \mathbf{k}'_{\perp}) &\equiv \frac{1}{4z_1} \int dk'^+ \int dk'^- \delta(k'^+ - P_1^+/z_1) \text{Tr}(\mathcal{D}\Gamma) \\ &= \frac{1}{4z_1} \int \frac{d\eta^- d^2\boldsymbol{\eta}_{\perp}}{(2\pi)^3} e^{i(P_1^+ \eta^- / z_1 - \mathbf{k}'_{\perp} \cdot \boldsymbol{\eta}_{\perp})} \sum_{X'} \int \frac{d^3 \mathbf{P}'_X}{(2\pi)^3 2E'_X} \times \\ &\quad \langle 0 | \Gamma \psi_i(0, \eta^-, \boldsymbol{\eta}_{\perp}) | P_1; X \rangle \langle P_1; X | \bar{\psi}_j(0) | 0 \rangle, \end{aligned} \quad (28)$$

At leading twist, and considering unpolarized or spinless hadrons, there are only two fragmentation functions:

$$\mathcal{D}^{[\gamma^+]} = D_1, \quad (29)$$

$$\mathcal{D}^{[i\sigma^{i+}\gamma_5]} = \frac{\epsilon_{\perp}^{ij} k'_{\perp j}}{m_1} H_1^{\perp}. \quad (30)$$

Here  $D_1$  is the ordinary unpolarized fragmentation function, whereas  $H_1^{\perp}$  is the Collins function, describing the fragmentation of transversely polarized quarks into an unpolarized hadron.

## 4 The hadronic tensor

Using the Fierz decomposition

$$\begin{aligned}
(\gamma^\mu)_{ij} (\gamma^\nu)_{kl} = & \frac{1}{4} \left\{ g^{\mu\nu} \left[ -(\gamma_\rho)_{il} (\gamma^\rho)_{kj} - (\gamma_\rho \gamma_5)_{il} (\gamma^\rho \gamma_5)_{kj} + \frac{1}{2} (i\sigma_{\alpha\beta} \gamma_5)_{il} (i\sigma^{\alpha\beta} \gamma_5)_{kj} \right] \right. \\
& + (\gamma^{\{\mu} \gamma^{\nu\}})_{il} (\gamma^{\nu\}})_{kj} + (\gamma^{\{\mu} \gamma_5)_{il} (\gamma^{\nu\}} \gamma_5)_{kj} - (i\sigma_\alpha^{\{\mu} \gamma_5)_{il} (i\sigma^{\nu\}} \gamma_5)_{kj} \\
& \left. + \epsilon^{\mu\nu\rho\sigma} [(\gamma_\rho)_{il} (\gamma_\sigma \gamma_5)_{kj} + (\gamma_\rho \gamma_5)_{il} (\gamma_\sigma)_{kj}] \right\} + \dots,
\end{aligned} \tag{31}$$

where the dots label terms that do not contribute to leading twist, we can re-express the hadronic tensor (22) as (we retain the leading-twist contributions only and return to the  $\gamma^* N$  collinear frame)

$$\begin{aligned}
W^{\mu\nu} = & 4z_1 \zeta_2 (2\pi)^3 \sum_a e_a^2 \int d^2 \mathbf{k}_\perp \int d^2 \mathbf{k}'_\perp \delta^2(\mathbf{k}_\perp - \mathbf{k}'_\perp - \mathbf{P}_{1\perp}/z_1) \\
& \times \left\{ -g_\perp^{\mu\nu} \mathcal{M}^{[\gamma^-]} \mathcal{D}^{[\gamma^+]} + g_\perp^{\mu\nu} \mathcal{M}^{[i\sigma_i^- \gamma_5]} \mathcal{D}^{[i\sigma^{i+} \gamma_5]} - \mathcal{M}^{[i\sigma^{\{\mu-} \gamma_5]} \mathcal{D}^{[i\sigma^{\nu\}} \gamma_5]} \right. \\
& \left. + i \epsilon_\perp^{\mu\nu} \mathcal{M}^{[\gamma^- \gamma_5]} \mathcal{D}^{[\gamma^+]} \right\}.
\end{aligned} \tag{32}$$

The first term in Eq. (32) couples the unpolarized fracture functions to the unpolarized fragmentation function  $D_1$ . The second and third terms involve the transversely polarized fracture functions and the Collins fragmentation function  $H_1^\perp$ . The last term represents the antisymmetric part of  $W^{\mu\nu}$ , which contributes to lepto-production with a polarized beam and couples the longitudinally polarized fracture functions to  $D_1$ .

Using Eqs. (25-27) and Eqs. (29-30), we get

$$\begin{aligned}
W^{\mu\nu} = & 4z_1 \zeta_2 (2\pi)^3 \sum_a e_a^2 \int d^2 \mathbf{k}_\perp \int d^2 \mathbf{k}'_\perp \delta^2(\mathbf{k}_\perp - \mathbf{k}'_\perp - \mathbf{P}_{1\perp}/z_1) \\
& \times \left\{ -g_\perp^{\mu\nu} \left[ \hat{u}_1 D_1 + \frac{\mathbf{P}_{2\perp} \times \mathbf{S}_\perp}{m_2} \hat{u}_{1T}^h D_1 + \frac{\mathbf{k}_\perp \times \mathbf{S}_\perp}{m_N} \hat{u}_{1T}^\perp D_1 + \frac{S_\parallel (\mathbf{k}_\perp \times \mathbf{P}_{2\perp})}{m_N m_2} \hat{u}_{1L}^{\perp h} D_1 \right] \right. \\
& - \frac{(S_\perp^{\{\mu} \epsilon_\perp^{\nu\}} k'_{\perp\rho} + k'_\perp{}^{\{\mu} \epsilon_\perp^{\nu\}} S_{\perp\rho})}{2m_1} \hat{t}_{1T} H_1^\perp \\
& - S_\parallel \frac{(P_{2\perp}^{\{\mu} \epsilon_\perp^{\nu\}} k'_{\perp\rho} + k'_\perp{}^{\{\mu} \epsilon_\perp^{\nu\}} P_{2\perp\rho})}{2m_1 m_2} \hat{t}_{1L}^h H_1^\perp \\
& - S_\parallel \frac{(k_\perp^{\{\mu} \epsilon_\perp^{\nu\}} k'_{\perp\rho} + k'_\perp{}^{\{\mu} \epsilon_\perp^{\nu\}} k_{\perp\rho})}{2m_1 m_N} \hat{t}_{1L}^\perp H_1^\perp \\
& + \frac{P_{2\perp} \cdot S_\perp (P_{2\perp}^{\{\mu} \epsilon_\perp^{\nu\}} k'_{\perp\rho} + k'_\perp{}^{\{\mu} \epsilon_\perp^{\nu\}} P_{2\perp\rho})}{2m_1 m_2^2} \hat{t}_{1T}^{hh} H_1^\perp \\
& + \frac{k_\perp \cdot S_\perp (k_\perp^{\{\mu} \epsilon_\perp^{\nu\}} k'_{\perp\rho} + k'_\perp{}^{\{\mu} \epsilon_\perp^{\nu\}} k_{\perp\rho})}{2m_1 m_N^2} \hat{t}_{1T}^{\perp h} H_1^\perp \\
& \left. + \frac{k_\perp \cdot S_\perp (P_{2\perp}^{\{\mu} \epsilon_\perp^{\nu\}} k'_{\perp\rho} + k'_\perp{}^{\{\mu} \epsilon_\perp^{\nu\}} P_{2\perp\rho}) + P_{2\perp} \cdot S_\perp (k_\perp^{\{\mu} \epsilon_\perp^{\nu\}} k'_{\perp\rho} + k'_\perp{}^{\{\mu} \epsilon_\perp^{\nu\}} k_{\perp\rho})}{2m_1 m_2 m_N} \hat{t}_{1T}^{\perp h} H_1^\perp \right\}
\end{aligned}$$



$$\begin{aligned}
& + \frac{P_{2\perp}^{\{\mu} k_{\perp}^{\nu\}} - g_{\perp}^{\mu\nu} P_{2\perp} \cdot k'_{\perp}}{m_1 m_2} \hat{t}_1^h H_1^{\perp} + \frac{k_{\perp}^{\{\mu} k'_{\perp}^{\nu\}} - g_{\perp}^{\mu\nu} k_{\perp} \cdot k'_{\perp}}{m_1 m_N} \hat{t}_1^{\perp} H_1^{\perp} \\
& + i \epsilon_{\perp}^{\mu\nu} \left[ S_{\parallel} \hat{l}_{1L} D_1 + \frac{\mathbf{P}_{2\perp} \cdot \mathbf{S}_{\perp}}{m_2} \hat{l}_{1T}^h D_1 \right. \\
& \left. + \frac{\mathbf{k}_{\perp} \cdot \mathbf{S}_{\perp}}{m_N} \hat{l}_{1T}^{\perp} D_1 + \frac{\mathbf{k}_{\perp} \times \mathbf{P}_{2\perp}}{m_N m_2} \hat{l}_1^{\perp h} D_1 \right] \Bigg\}. \tag{33}
\end{aligned}$$

The fully differential cross section for two-hadron production is obtained by contracting the hadronic tensor (33) with the leptonic tensor, Eqs. (10, 11). The final expression is extremely complicated and will be reported elsewhere. In the following we will focus on double lepto-production integrated over the transverse momentum either of the hadron produced in the TFR, or of the hadron produced in the CFR.

## 5 Double hadron lepto-production integrated over $\mathbf{P}_{2\perp}$

If we integrate the fracture matrix over  $\mathbf{P}_{2\perp}$  we are left with eight  $k_{\perp}$ -dependent fracture functions:

$$\int d^2 \mathbf{P}_{2\perp} \mathcal{M}^{[\gamma^-]} = u_1 + \frac{\mathbf{k}_{\perp} \times \mathbf{S}_{\perp}}{m_N} u_{1T}^{\perp}, \tag{34}$$

$$\int d^2 \mathbf{P}_{2\perp} \mathcal{M}^{[\gamma^- \gamma_5]} = S_{\parallel} l_{1L} + \frac{\mathbf{k}_{\perp} \cdot \mathbf{S}_{\perp}}{m_N} l_{1T}, \tag{35}$$

$$\begin{aligned}
\int d^2 \mathbf{P}_{2\perp} \mathcal{M}^{[i \sigma^{i-} \gamma_5]} &= S_{\perp}^i t_{1T} + \frac{S_{\parallel} k_{\perp}^i}{m_N} t_{1L}^{\perp} + \frac{k_{\perp}^i (\mathbf{k}_{\perp} \cdot \mathbf{S}_{\perp})}{m_N^2} t_{1T}^{\perp} + \frac{\epsilon_{\perp}^{ij} k_{\perp j}}{m_N} t_1^{\perp} \\
&= S_{\perp}^i t_1 + \frac{S_{\parallel} k_{\perp}^i}{m_N} t_{1L}^{\perp} + \frac{(k_{\perp}^i k_{\perp}^j - \frac{1}{2} \mathbf{k}_{\perp}^2 \delta_{ij}) S_{\perp}^j}{m_N^2} t_{1T}^{\perp} + \frac{\epsilon_{\perp}^{ij} k_{\perp j}}{m_N} t_1^{\perp}, \tag{36}
\end{aligned}$$

where  $t_1 \equiv t_{1T} + (\mathbf{k}_{\perp}^2/2m_N^2) t_{1T}^{\perp}$ . We have removed the hat to denote the  $\mathbf{P}_{2\perp}$ -integrated fracture functions:

$$u_1(x_B, \mathbf{k}_{\perp}^2, \zeta_2) = \int d^2 \mathbf{P}_{2\perp} \hat{u}_1, \tag{37}$$

$$u_{1T}^{\perp}(x_B, \mathbf{k}_{\perp}^2, \zeta_2) = \int d^2 \mathbf{P}_{2\perp} \left\{ \hat{u}_{1T}^{\perp} + \frac{m_N}{m_2} \frac{\mathbf{k}_{\perp} \cdot \mathbf{P}_{2\perp}}{\mathbf{k}_{\perp}^2} \hat{u}_{1T}^h \right\}, \tag{38}$$

$$l_{1L}(x_B, \mathbf{k}_{\perp}^2, \zeta_2) = \int d^2 \mathbf{P}_{2\perp} \hat{l}_{1L}, \tag{39}$$

$$l_{1T}(x_B, \mathbf{k}_{\perp}^2, \zeta_2) = \int d^2 \mathbf{P}_{2\perp} \left\{ \hat{l}_{1T}^{\perp} + \frac{m_N}{m_2} \frac{\mathbf{k}_{\perp} \cdot \mathbf{P}_{2\perp}}{\mathbf{k}_{\perp}^2} \hat{l}_{1T}^h \right\}, \tag{40}$$

$$t_1(x_B, \mathbf{k}_{\perp}^2, \zeta_2) = \int d^2 \mathbf{P}_{2\perp} \left\{ \hat{t}_{1T} + \frac{\mathbf{k}_{\perp}^2}{2m_N^2} \hat{t}_{1T}^{\perp} + \frac{\mathbf{P}_{2\perp}^2}{2m_2^2} \hat{t}_{1T}^{hh} \right\}. \tag{41}$$

$$t_{1L}^{\perp}(x_B, \mathbf{k}_{\perp}^2, \zeta_2) = \int d^2 \mathbf{P}_{2\perp} \left\{ \hat{t}_{1L}^{\perp} + \frac{m_N}{m_2} \frac{\mathbf{k}_{\perp} \cdot \mathbf{P}_{2\perp}}{\mathbf{k}_{\perp}^2} \hat{t}_{1L}^h \right\}, \tag{42}$$

$$t_{1T}^\perp(x_B, \mathbf{k}_\perp^2, \zeta_2) = \int d^2 \mathbf{P}_{2\perp} \left\{ \hat{t}_{1T}^{\perp\perp} + \frac{m_N^2}{m_2^2} \frac{2(\mathbf{k}_\perp \cdot \mathbf{P}_{2\perp})^2 - \mathbf{k}_\perp^2 \mathbf{P}_{2\perp}^2}{(\mathbf{k}_\perp^2)^2} \hat{t}_{1T}^{hh} \right\}, \quad (43)$$

$$t_1^\perp(x_B, \mathbf{k}_\perp^2, \zeta_2) = \int d^2 \mathbf{P}_{2\perp} \left\{ \hat{t}_1^\perp + \frac{m_N}{m_2} \frac{\mathbf{k}_\perp \cdot \mathbf{P}_{2\perp}}{\mathbf{k}_\perp^2} \hat{t}_1^h \right\}. \quad (44)$$

By virtue of the sum rules derived in Ref. [1], these fracture functions are directly related to the eight leading-twist distribution functions by

$$\sum_h \int d\zeta_2 \zeta_2 u_1(x_B, \mathbf{k}_\perp^2, \zeta_2) = (1 - x_B) f_1(x_B, \mathbf{k}_\perp^2), \quad (45)$$

$$\sum_h \int d\zeta_2 \zeta_2 u_{1T}^\perp(x_B, \mathbf{k}_\perp^2, \zeta_2) = -(1 - x_B) f_{1T}^\perp(x_B, \mathbf{k}_\perp^2), \quad (46)$$

$$\sum_h \int d\zeta_2 \zeta_2 l_{1L}(x_B, \mathbf{k}_\perp^2, \zeta_2) = (1 - x_B) g_{1L}(x_B, \mathbf{k}_\perp^2), \quad (47)$$

$$\sum_h \int d\zeta_2 \zeta_2 l_{1T}(x_B, \mathbf{k}_\perp^2, \zeta_2) = (1 - x_B) g_{1T}(x_B, \mathbf{k}_\perp^2), \quad (48)$$

$$\sum_h \int d\zeta_2 \zeta_2 t_1(x_B, \mathbf{k}_\perp^2, \zeta_2) = (1 - x_B) h_1(x_B, \mathbf{k}_\perp^2), \quad (49)$$

$$\sum_h \int d\zeta_2 \zeta_2 t_{1L}^\perp(x_B, \mathbf{k}_\perp^2, \zeta_2) = (1 - x_B) h_{1L}^\perp(x_B, \mathbf{k}_\perp^2), \quad (50)$$

$$\sum_h \int d\zeta_2 \zeta_2 t_{1T}^\perp(x_B, \mathbf{k}_\perp^2, \zeta_2) = (1 - x_B) h_{1T}^\perp(x_B, \mathbf{k}_\perp^2), \quad (51)$$

$$\sum_h \int d\zeta_2 \zeta_2 t_1^\perp(x_B, \mathbf{k}_\perp^2, \zeta_2) = -(1 - x_B) h_1^\perp(x_B, \mathbf{k}_\perp^2). \quad (52)$$

Notice that, among the 16 fracture functions listed in Eqs. (25-27), the three functions with double superscript  $\perp h$ , i.e.  $\hat{u}_{1L}^{\perp h}$ ,  $\hat{l}_1^{\perp h}$  and  $\hat{h}_{1T}^{\perp h}$ , which measure correlations involving both the quark and the hadron transverse momenta, have no distribution function counterpart and disappear once the integration over any of the two transverse momenta is performed. In particular,  $\hat{l}_1^{\perp h}$ , which describes longitudinally polarized quarks inside an unpolarized nucleon, is not probed in single hadron lepto-production, whereas it gives rise to a beam spin asymmetry in (unintegrated) two-hadron lepto-production [6, 7].

The  $\mathbf{P}_{2\perp}$ -integrated hadronic tensor reads

$$\begin{aligned} \int d^2 \mathbf{P}_{2\perp} W^{\mu\nu} &= 4z_1 \zeta_2 (2\pi)^3 \sum_a e_a^2 \int d^2 \mathbf{k}_\perp \int d^2 \mathbf{k}'_\perp \delta^2(\mathbf{k}_\perp - \mathbf{k}'_\perp - \mathbf{P}_{1\perp}/z_1) \\ &\times \left\{ -g_\perp^{\mu\nu} \left[ u_1 D_1 + \frac{\mathbf{k}_\perp \times \mathbf{S}_\perp}{m_N} u_{1T}^\perp D_1 \right] \right. \\ &\quad - \frac{(S_\perp^{\{\mu} \epsilon_\perp^{\nu\}\rho} k'_{\perp\rho} + k'_\perp{}^{\{\mu} \epsilon_\perp^{\nu\}\rho} S_{\perp\rho})}{2m_1} t_{1T} H_1^\perp \\ &\quad \left. - S_\parallel \frac{(k_\perp^{\{\mu} \epsilon_\perp^{\nu\}\rho} k'_{\perp\rho} + k'_\perp{}^{\{\mu} \epsilon_\perp^{\nu\}\rho} k_{\perp\rho})}{2m_1 m_N} t_{1L}^\perp H_1^\perp \right\} \end{aligned}$$

$$\begin{aligned}
& + \frac{k_{\perp} \cdot S_{\perp} \left( k_{\perp}^{\{\mu} \epsilon_{\perp}^{\nu\}\rho} k'_{\perp\rho} + k'_{\perp}^{\{\mu} \epsilon_{\perp}^{\nu\}\rho} k_{\perp\rho} \right)}{2m_1 m_N^2} t_{1T}^{\perp} H_1^{\perp} \\
& + \frac{k_{\perp}^{\{\mu} k'_{\perp}^{\nu\}} - g_{\perp}^{\mu\nu} k_{\perp} \cdot k'_{\perp}}{m_1 m_N} t_1^{\perp} H_1^{\perp} \\
& + i \epsilon_{\perp}^{\mu\nu} \left[ S_{\parallel} l_{1L} D_1 + \frac{\mathbf{k}_{\perp} \cdot \mathbf{S}_{\perp}}{m_N} l_{1T} D_1 \right] \Bigg\}. \tag{53}
\end{aligned}$$

This hadronic tensor is perfectly analogous to the one describing single-hadron leptonproduction in the CFR [2], the correspondence being: Fracture Functions  $\Rightarrow$  Distribution Functions. Thus we can use the procedure of Ref. [2] to contract the hadronic tensor and the leptonic tensor. The final expression of the cross section is

$$\begin{aligned}
\frac{d\sigma}{dx_B dy dz_1 d\zeta_2 d\phi_1 dP_{1\perp}^2 d\phi_S} &= \frac{\alpha_{\text{em}}^2}{x_B y Q^2} \left\{ \left(1 - y + \frac{y^2}{2}\right) \mathcal{F}_{UU,T} + (1 - y) \cos 2\phi_1 \mathcal{F}_{UU}^{\cos 2\phi_1} \right. \\
&+ S_{\parallel} (1 - y) \sin 2\phi_1 \mathcal{F}_{UL}^{\sin 2\phi_1} + S_{\parallel} \lambda_{\ell} y \left(1 - \frac{y}{2}\right) \mathcal{F}_{LL} \\
&+ S_{\perp} \left(1 - y + \frac{y^2}{2}\right) \sin(\phi_1 - \phi_S) \mathcal{F}_{UT}^{\sin(\phi_1 - \phi_S)} \\
&+ S_{\perp} (1 - y) \sin(\phi_1 + \phi_S) \mathcal{F}_{UT}^{\sin(\phi_1 + \phi_S)} + S_{\perp} (1 - y) \sin(3\phi_1 - \phi_S) \mathcal{F}_{UT}^{\sin(3\phi_1 - \phi_S)} \\
&\left. + S_{\perp} \lambda_{\ell} y \left(1 - \frac{y}{2}\right) \cos(\phi_1 - \phi_S) \mathcal{F}_{LT}^{\cos(\phi_1 - \phi_S)} \right\} \tag{54}
\end{aligned}$$

where the structure functions are given at leading twist by ( $\hat{\mathbf{P}}_1 \equiv \mathbf{P}_{1\perp}/|\mathbf{P}_{1\perp}|$ )

$$\mathcal{F}_{UU,T} = \mathcal{C} [u_1 D_1], \tag{55}$$

$$\mathcal{F}_{UU}^{\cos 2\phi_1} = \mathcal{C} \left[ \frac{2(\hat{\mathbf{P}}_1 \cdot \mathbf{k}_{\perp})(\hat{\mathbf{P}}_1 \cdot \mathbf{k}'_{\perp}) - \mathbf{k}_{\perp} \cdot \mathbf{k}'_{\perp}}{m_N m_1} t_1^{\perp} H_1^{\perp} \right], \tag{56}$$

$$\mathcal{F}_{UL}^{\sin 2\phi_1} = \mathcal{C} \left[ -\frac{2(\hat{\mathbf{P}}_1 \cdot \mathbf{k}_{\perp})(\hat{\mathbf{P}}_1 \cdot \mathbf{k}'_{\perp}) - \mathbf{k}_{\perp} \cdot \mathbf{k}'_{\perp}}{m_N m_1} t_{1L}^{\perp} H_1^{\perp} \right], \tag{57}$$

$$\mathcal{F}_{LL} = \mathcal{C} [l_{1L} D_1], \tag{58}$$

$$\mathcal{F}_{UT}^{\sin(\phi_1 - \phi_S)} = \mathcal{C} \left[ \frac{\hat{\mathbf{P}}_1 \cdot \mathbf{k}_{\perp}}{m_N} u_{1T}^{\perp} D_1 \right], \tag{59}$$

$$\mathcal{F}_{UT}^{\sin(\phi_1 + \phi_S)} = \mathcal{C} \left[ -\frac{\hat{\mathbf{P}}_1 \cdot \mathbf{k}'_{\perp}}{m_1} t_1 H_1^{\perp} \right], \tag{60}$$

$$\mathcal{F}_{UT}^{\sin(3\phi_1 - \phi_S)} = \mathcal{C} \left[ \frac{2(\hat{\mathbf{P}}_1 \cdot \mathbf{k}'_{\perp})(\mathbf{k}_{\perp} \cdot \mathbf{k}'_{\perp}) + \mathbf{k}_{\perp}^2 (\hat{\mathbf{P}}_1 \cdot \mathbf{k}'_{\perp}) - 4(\hat{\mathbf{P}}_1 \cdot \mathbf{k}_{\perp})^2 (\hat{\mathbf{P}}_1 \cdot \mathbf{k}'_{\perp})}{2m_N^2 m_1} t_{1T}^{\perp} H_1^{\perp} \right], \tag{61}$$

$$\mathcal{F}_{LT}^{\cos(\phi_1 - \phi_S)} = \mathcal{C} \left[ \frac{\hat{\mathbf{P}}_1 \cdot \mathbf{k}_{\perp}}{m_N} l_{1T} D_1 \right], \tag{62}$$

with the following notation for the transverse momenta convolutions

$$\mathcal{C} [wuD] = \sum_a e_a^2 x_B \int d^2 \mathbf{k}_{\perp} \int d^2 \mathbf{k}'_{\perp} \delta^2(\mathbf{k}_{\perp} - \mathbf{k}'_{\perp} - \mathbf{P}_{1\perp}/z_1)$$

$$\times w(\mathbf{k}_\perp, \mathbf{k}'_\perp) u^a(x_B, \mathbf{k}_\perp^2, \zeta_2) D^a(z_1, \mathbf{k}'_\perp{}^2). \quad (63)$$

## 6 Double hadron lepto-production integrated over $\mathbf{P}_{1\perp}$

If one integrates the hadronic tensor over the transverse momentum  $\mathbf{P}_{1\perp}$  of the hadron produced in the CFR, the two integrations over  $\mathbf{k}_\perp$  and  $\mathbf{k}'_\perp$  in Eq. (32) disentangle and can be performed separately.

Integrating the fragmentation matrix over  $\mathbf{k}'_\perp$ , only one fragmentation function,  $D_1$ , survives, which couples to the unpolarized and the longitudinally polarized fracture functions. The relevant fracture matrix projections integrated over  $\mathbf{k}_\perp$  are [1]

$$\int d^2\mathbf{k}_\perp \mathcal{M}^{[\gamma^-]} = \tilde{u}_1(x_B, \zeta_2, \mathbf{P}_{2\perp}^2) + \frac{\mathbf{P}_{2\perp} \times \mathbf{S}_\perp}{m_2} \tilde{u}_{1T}^h(x_B, \zeta_2, \mathbf{P}_{2\perp}^2), \quad (64)$$

$$\int d^2\mathbf{k}_\perp \mathcal{M}^{[\gamma^- \gamma_5]} = S_\parallel \tilde{l}_{1L}(x_B, \zeta_2, \mathbf{P}_{2\perp}^2) + \frac{\mathbf{P}_{2\perp} \cdot \mathbf{S}_\perp}{m_2} \tilde{l}_{1T}^h(x_B, \zeta_2, \mathbf{P}_{2\perp}^2), \quad (65)$$

with (a tilde denotes fracture functions integrated over the quark transverse momentum)

$$\tilde{u}_1(x_B, \zeta_2, \mathbf{P}_{2\perp}^2) = \int d^2\mathbf{k}_\perp \hat{u}_1, \quad (66)$$

$$\tilde{u}_{1T}^h(x_B, \zeta_2, \mathbf{P}_{2\perp}^2) = \int d^2\mathbf{k}_\perp \left\{ \hat{u}_{1T}^h + \frac{m_2}{m_N} \frac{\mathbf{k}_\perp \cdot \mathbf{P}_{2\perp}}{\mathbf{P}_{2\perp}^2} \hat{u}_{1T}^\perp \right\}, \quad (67)$$

$$\tilde{l}_{1L}(x_B, \zeta_2, \mathbf{P}_{2\perp}^2) = \int d^2\mathbf{k}_\perp \hat{l}_{1L}, \quad (68)$$

$$\tilde{l}_{1T}^h(x_B, \zeta_2, \mathbf{P}_{2\perp}^2) = \int d^2\mathbf{k}_\perp \left\{ \hat{l}_{1T}^h + \frac{m_2}{m_N} \frac{\mathbf{k}_\perp \cdot \mathbf{P}_{2\perp}}{\mathbf{P}_{2\perp}^2} \hat{l}_{1T}^\perp \right\}. \quad (69)$$

The final result for the cross section is

$$\begin{aligned} \frac{d\sigma}{dx_B dy dz_1 d\zeta_2 d\phi_2 dP_{2\perp}^2 d\phi_S} &= \frac{\alpha_{\text{em}}^2}{y Q^2} \left\{ \left( 1 - y + \frac{y^2}{2} \right) \right. \\ &\times \sum_a e_a^2 \left[ \tilde{u}_1(x_B, \zeta_2, \mathbf{P}_{2\perp}^2) - |\mathbf{S}_\perp| \frac{|\mathbf{P}_{2\perp}|}{m_2} \tilde{u}_{1T}^h(x_B, \zeta_2, \mathbf{P}_{2\perp}^2) \sin(\phi_2 - \phi_S) \right] \\ &+ \lambda y \left( 1 - \frac{y}{2} \right) \sum_a e_a^2 \left[ S_\parallel \tilde{l}_{1L}(x_B, \zeta_2, \mathbf{P}_{2\perp}^2) \right. \\ &\left. \left. + |\mathbf{S}_\perp| \frac{|\mathbf{P}_{2\perp}|}{m_2} \tilde{l}_{1T}^h(x_B, \zeta_2, \mathbf{P}_{2\perp}^2) \cos(\phi_2 - \phi_S) \right] \right\} D_1(z_1). \end{aligned} \quad (70)$$

As in the case of single-hadron production [1], there is a Sivers-type modulation  $\sin(\phi_2 - \phi_S)$ , but no Collins-type effect.

## 7 Conclusions and perspectives

We have considered the double production of unpolarized hadrons in deep inelastic scattering processes, with one hadron produced in the current fragmentation region and one in the target fragmentation region. We have combined the fracture function formalism, which describes the hadron production in the TFR [8, 9] and the fragmentation function formalism, which describes the hadron production in the CFR. Target polarization and the transverse motions, of quarks inside the parent hadron, and in the fragmentation process, have been taken into account. TMD factorization has been assumed.

This paper completes a previous one [1] in which the formalism of polarized and transverse momentum dependent fracture functions was introduced to describe single hadron production in the TFR. The combined observation of one hadron in the TFR and one hadron in the CFR allows access to a class of chiral-odd fracture functions which cannot contribute to single hadron production.

The general result for the hadronic tensor involving all fracture and fragmentation functions is given in Eq. (33), and the corresponding cross section can be obtained by inserting this result into Eq. (8) and using Eqs. (10) and (11). We have not presented explicitly the final expression, which is rather cumbersome, but have considered more realistic cases in which one only measures the longitudinal component of one of the final hadrons, integrating over its transverse momentum. Such results are given in Eqs. (54) and (70).

Eq. (54), which refers to the case in which one detects the three momentum  $(z_1, \mathbf{P}_{1\perp})$  of one hadron in the CFR (like in the usual SIDIS) and the longitudinal momentum fraction  $\zeta_2$  of another hadron in the TFR, has the same structure as the familiar cross section for single hadron production in the CFR, with the role of the distribution functions (TMDs) replaced by the fracture functions. Therefore, it has the same potentiality, for measuring the  $\mathbf{P}_{2\perp}$ -integrated fracture functions, as the usual CFR SIDIS for measuring the TMDs.

Eq. (70), which refers to the case in which one detects the three momentum  $(\zeta_2, \mathbf{P}_{2\perp})$  of one hadron in the TFR and the longitudinal momentum fraction  $z_1$  of another hadron in the CFR, has the same structure as the cross section for the single hadron production in the TFR, Eq. (51) of Ref. [1], with the addition of an integrated fragmentation function. The presence of these fragmentation functions in principle allows to perform the quark flavor decomposition of quark transverse momentum integrated fracture functions as it is done for distribution functions in SIDIS in the CFR.

Phenomenological analyses of SIDIS data, based on the results presented here and in Ref. [1], could confirm our full understanding of the mechanism of hadron production in lepton-nucleon interactions. The observation and measurement of the predicted azimuthal dependences in the TFR would allow the extraction of the fracture functions, similarly to what is being done for TMDs in the CFR.

The clear disentanglement of effects observed in the two regions, TFR and CFR, is crucial for an unambiguous interpretation of the data. Although some first results might be available soon from JLab and COMPASS, we think that the ideal experiments to test the fracture function factorization and measure these new functions in SIDIS, are those being discussed in the international community and planned at future Electron Ion or Electron Nucleon Colliders (EIC/ENC).

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## References

- [1] M. Anselmino, V. Barone and A. Kotzinian, *Phys. Lett.* **B699** (2011) 108, arXiv:1102.4214 [hep-ph].
- [2] P.J. Mulders and R.D. Tangerman, *Nucl. Phys.* **B461** (1996) 197, hep-ph/9510301.
- [3] V. Barone, A. Drago and P.G. Ratcliffe, *Phys. Rep.* **359** (2002) 1, hep-ph/0104283.
- [4] A.V. Belitsky, X. Ji and F. Yuan, *Nucl. Phys.* **B656** (2003) 165, hep-ph/0208038.
- [5] C.J. Bomhof, P.J. Mulders and F. Pijlman, *Eur. Phys. J.* **C47** (2006) 147, hep-ph/0601171.
- [6] M. Anselmino, V. Barone and A. Kotzinian, paper in preparation.
- [7] A. Kotzinian, M. Anselmino and V. Barone, arXiv:1107.2292 [hep-ph].
- [8] L. Trentadue and G. Veneziano, *Phys. Lett.* **B323** (1994) 201.
- [9] M. Grazzini, L. Trentadue and G. Veneziano, *Nucl. Phys.* **B519** (1998) 394, hep-ph/9709452.